MATHEMATICAL ANALYSIS OF NON-STATIONARY PROCESSES OF INFILTRATION INTO UNSATURATED SOIL

MATEMATISK ANALYS AV ICKE-STATIONÄR INFILTRATION I OMÄTTAD JORD


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Abstract
Theoretical studies of the movement of moisture in an unsaturated zone are associated with the analysis of the Richards equations. The corresponding mathematical problems are designed to answer practical questions related to the replenishment of groundwater reserves, the degree of their protection from pollution coming from the surface, and the needs of agriculture. The value of surface recharge appears in the problems as a boundary condition at the upper boundary of the soil. It depends on weather conditions, which leads to significant non-stationarity of the infiltration processes. At the same time, for a broad class of surface feeding modes, fluctuations of moisture flows are smoothed out with depth. The depth at which fluctuations in the moisture flow become negligible and the flow becomes almost stationary is of interest for many practical applications. In this work, explicit formulas are derived analytically for this depth, expressing its value through the filtration parameters of the medium and the characteristics of the mode of moisture ingress into the soil.

Sammanfattning
Teoretiska studier av fuktrörelser i den omättaade zonen kan kopplas till Richards ekvation.Korresponderande matematiska problem besvarar praktiska frågor relaterade till tillflöde till grundvattenreservoarer och hur de skyddas från ytföroreningenar, samt jordbruksnäringens vattenbehov. Randvillkoret på övre gränsen av markytan utgörs av vattenmängden som distribueras till densamma. Således varierar detta med vädret, vars variationer leder till signifikanta variationer i infiltrationsprocesserna. Djupet på vilket fluktuationer i fuktrörelserna blir försumbara har många intressanta praktiska tillämpningar. I detta arbete härleds formler för detta djup analytiskt och dess värde uttrycks genom de paramterar som beskriver markens infiltrationsegenskaper och det sätt på vilket vattnet distribueras; lågintensivt dagsregn, häftig åskskur, med mera.

Introduction

The unsteady nature of precipitation results in uneven infiltration of atmospheric moisture into the unsaturated soil zone. This leaves an imprint on the movement of moisture from the surface of the earth to groundwater. The distribution of moisture in the soil and its flows in depth and time determines such parameters of these processes as the amount of groundwater recharge and the time it takes for moisture to reach their surface. The research on the unsteady motion of water in the unsaturated ground has a rich history. The theoretical part of them is connected with the analysis of the Richards equation and its various modifications. The corresponding mathematical problems are designed to answer practical questions related to the replenishment of groundwater reserves, the degree of their protection from pollution coming from the surface, and the needs of agriculture. In this article, the mathematical problem of vertical infiltration of moisture during its impulse flow through the upper boundary of the soil is investigated analytically.

The amount of surface recharge, which is used as a boundary condition at the upper boundary of the soil, is determined not only by the intensity of atmospheric precipitation but also depends on the microrelief, soil condition, and weather conditions. Unfortunately, direct measurement of this quantity is complex, and in field conditions, it is impossible. Therefore, the quantitative relationship between it and the precipitation regime is unknown. At the same time, if the problem of the impulse input of moisture into the soil is associated with infiltration due to a single rainfall, then at the qualitative level it is evident that the duration of the impulse should be the longer, the longer the rain and the amplitude of the supply should increase with the increase in its intensity. The paper considers instances of pressure and non-pressure pulses supply. The latter can be interpreted as a consequence of heavy precipitation with the formation of puddles on the earth's surface.

To assess the protection of groundwater, an essential characteristic of the infiltration process is the rate of movement of moisture in the ground. It determines the time it takes for moisture particles and pollutants dissolved to reach groundwater's surface. For stationary modes of infiltration, in many textbooks, monographs, and manuals on the theory of filtration (Bindeman, 1963, Goldberg, 1984, Belousova, 2006), a simple formula of the form is derived for this quantity:

$$V = Const \cdot R^{\alpha} \tag{1}$$

where *Const* and $\alpha, 0 < \alpha < 1$ are constants and depend on the filtration parameters of the soil, and R is the constant value of the feed entering the soil from the surface. If the infiltration process is essentially non-stationary, then formula (1) becomes inapplicable. Nevertheless, if the fluctuations of moisture fluxes are smoothed out with depth in the process of infiltration, then the unsteady movement of moisture manifests itself only in the near-surface zone of the soil. Outside this zone (provided that its bottom is located above the surface of the groundwater), the speed of moisture movement can be estimated using stationary formulas.

In this article, the influence of the non-stationarity of the magnitude of the surface supply $R=R(t)$ of the pulsed type on the distribution of moisture and moisture flows over depth and time is studied. The results and conclusions are substantiated by constructing exact and approximate analytical solutions and are also illustrated numerically.

An essential qualitative conclusion to be drawn from the analysis of the solutions is that the pulsed mode, the infiltration of moisture and moisture distribution flow with increasing depth and is aligned closer to stationary. In this case, two physical mechanisms stabilise the flow: capillary dissipation and nonlinear dispersion of irregularities in the moisture profile. The depth below which the flow is stabilised and can be considered stationary depends on the parameters characterising the mode of moisture intake and the filtration properties of the soil. The main goal of our research is to calculate this depth. In this work, simple explicit formulas were derived for it, by which it is possible to estimate this depth for various values of the parameters. The formulas also contain the threshold value of the fluctuation amplitude, below which they can be considered insignificant.

The problem of water infiltration into the soil required to determine the dependence of the water saturation θ ($0 \le \theta \le 1$)and moisture flow q on time t and depth z. These functions satisfy the mass conservation equation and the Darcy relation

$$m\frac{\partial \theta}{\partial t} + \frac{\partial q}{\partial z} = 0, \qquad q = Kk(\theta)\left(1 - \frac{\partial p}{\partial z}\right), \qquad (2)$$

where m- porosity, K- coefficient of filtration rate in saturated soil, $k(\theta)$ - relative permeability, $p = p(t,z)$ - water pressure. Here and everywhere in what follows, it is assumed that the value p is measured from atmospheric pressure and is normalised to the specific gravity of water, and the axis z is directed vertically downward.

Further in the article, the effects associated with the influence of residual water saturation on the infiltration process will not be considered. In this case, the relative permeability $k(\theta)$ for liquids wetting the soil is usually given by a power-law formula of the form $k(\theta) = \theta^{\beta}$ with some indicator $\beta > 1$ (its most popular value is $\beta = 3$).

To complete equations (2), it is necessary to set the relationship between water saturation and pressure. In the Richards model (see J. Bear, 1971, Ch. 9), it is assumed that in the water-saturated flow zone $\theta = 1$ and $p \ge 0$, and in the unsaturated zone $\theta < 1$ and $p = -P^c(\theta)$, where $P^c(\theta)$ is an empirical function decreasing by θ, called capillary pressure. For the capillary function $P^c(\theta)$ infiltration models, various explicit dependencies are used, for example, the Brooks and Corey power formula:

$$P^c(\theta) = h\theta^{-\gamma}, \qquad (3)$$

where h- dimensional constant, by order of magnitude corresponding to the typical height of the capillary rise of water in the soil pores, and the index $\gamma > 0$ depends on the particle size distribution of the porous medium.

Equation (2) have a family of simple stationary solutions of the form

$$\theta \equiv \theta_0, \quad q \equiv R_0 = K\theta_0^{\beta}, \quad p \equiv 0, \qquad (4)$$

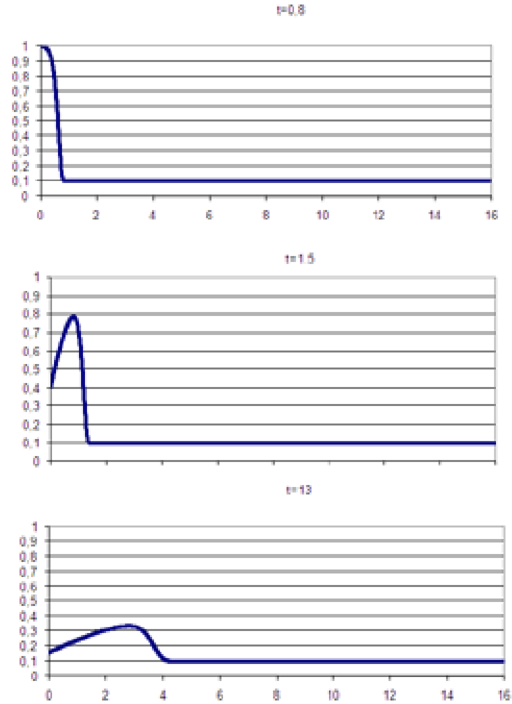

Fig. 1. *Calculated profiles of water saturation in the non-stationary infiltration problem with pulsed admission of moisture in the ground (h = 0.2 m)*

describing the infiltration of moisture at a rate constant in-depth and time, calculated by a formula $V = K\theta_0^{\beta-1}/m = K^{1/\beta}R_0^{1-1/\beta}/m$ that coincides with equality (1).

Consider for the original nonlinear equation (2) the problem with time-dependent infiltration $q(t,0) = R(t)$ into the soil of infinite depth $0 < z < +\infty$. As a condition at an infinite depth, we assume that the solution tends to a uniform flow (4) at $z \to +\infty$. Let the same value of water saturation θ_0 be the initial condition $\theta(0,z) \equiv \theta_0$. Figures 1 and 2 show the results of a numerical calculation of water saturation profiles $\theta = \theta(t,z)$ over depth at different points in time for infiltration $R(t)$ of an impulse type.

$$R(t) = R_0 = K\theta_0^{\beta} \quad \text{at} \quad t > T \quad \text{and} \qquad (5)$$
$$R(t) = R_1 > R_0 \quad \text{at} \quad 0 < t < T.$$

In the examples considered for the parameter h that characterizes the scale of capillary forces, the values of 0.05 m and 0.2 m are taken. The other parameters of the problem are given as follows

$T = 1$ day, $R_0 = 0.0002$ m/day, $R_1 = 0.2$ m/day, $K = 0.2$ m/day, $m = 0.3$, $\beta = 3$, $\gamma = 1$.

Here, the impulse mode of moisture ingress into the soil is characterised by the duration T and amplitudes R_1, R_0 impulse and background, respectively. With the considered infiltration mode, the distribution of moisture and moisture flow levels out over time and approaches the stationary one. The characteristic value of the depth below which the flow is stabilised and can be considered stationary depends on the parameters characterising the moisture intake mode and the soil's filtration properties. In this case, two physical mechanisms stabilise the flow: capillary dissipation and nonlinear dispersion of irregularities in the moisture profile. If both mechanisms work together, then it is impossible to describe the stabilisation of solutions in an explicit form, and only numerical experiments remain available. In this case, the problem of impulse infiltration contains too many parameters for the numerical estimates of the desired stabilisation depth to be universal. On the other hand, the Richards equations have neither exact self-similar solutions suitable for this problem nor, moreover, general expressions that make it possible to find a solution for an infiltration pulse *R(t)* of a more or less arbitrary shape. This makes the problem of constructing analytical estimates for the desired stabilisation depth nontrivial.

If one of the factors, nonlinearity or capillarity, dominates, while the other can be neglected, then the problem's solution is in a form convenient for its study. Considering the influence of only capillary forces on the stabilisation of the flow for equations (2), (3) can be investigated through their linearisation. A necessary condition, in this case, is the ratio $|\theta(t,z) - \theta_0| << \theta_0$ (for $\theta_0 = 0$, i.e., for infiltration into dry soil, the Richards equations are not linearised). Usually, the experimental accuracy of determining the filtration values is low;

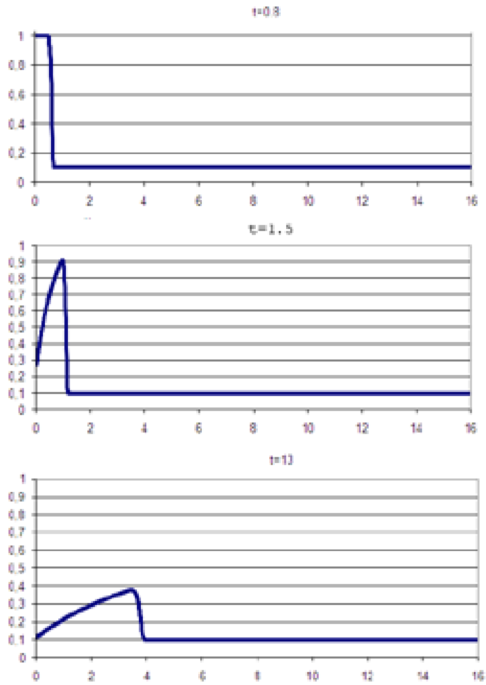

Fig. 2. *Water saturation profiles in the problem of non-stationary infiltration with impulse moisture inflow into the soil (h = 0.05 m)*

therefore, in practice, when this condition is met, it is difficult to distinguish a steady flow from a non-steady flow. Because of this, the applied value of the linear approximation is small. It nevertheless plays a role in theoretical questions, for example, in studying problems of the stability of solutions. If the capillary forces are neglected in the Richards equations, then the order of these equations will decrease and the infiltration problem is solved in an explicit, albeit cumbersome way for a broad class of given moisture fluxes *R(t)*.

Below, we investigate the infiltration problem for linearised and capillary-free problems and propose a method for adding the effects of capillary dissipation and nonlinear scattering for problems when these factors act together. As a result, formulas are derived to estimate the depth of the influence of unsteady power supply fluctuations on the surface.

Exact analytical solution of the infiltration problem in the linearised setting

Linearisation of the problem of impulse infiltration near the stationary solution (4) leads to the following relations for fluctuations in water saturation and flow:

$$m\frac{\partial\theta'}{\partial t}+\frac{\partial q'}{\partial z}=0,\quad q'=mV_0\left(\theta'-\sigma\frac{\partial\theta'}{\partial z}\right), \tag{6}$$

$$\theta'(0,z)=0,\qquad q'(t,0)=r(t)=R(t)-R_0>0,$$

where
$V_0=\beta K\theta_0^{\beta-1}/m,\ \sigma=\theta_0(-dP^c(\theta_0)/d\theta)/\beta=\gamma h\theta_0^{-\chi}/\beta.$

Impulse modes of moisture supply to the soil correspond to such nonnegative fluctuations of supply $r(t)$, which vanish starting from a specific moment $t=T$.

The general solution of this linear problem can be represented as Duhamel's integral:

$$q'(t,z)=\frac{V_0}{\sigma}\int_0^t r(\tau)Q\left(\frac{V_0(t-\tau}{\sigma},\frac{z}{\sigma}\right)d\tau, \tag{7}$$

$$\theta'(t,z)=\frac{1}{m\sigma}\int_0^t r(\tau)\Theta\left(\frac{V_0(t-\tau}{\sigma},\frac{z}{\sigma}\right)d\tau,$$

where dimensionless functions of dimensionless variables $Q(t,z)$ and $\Theta(t,z)$ are solutions to the problem

$$\frac{\partial\Theta}{\partial t}Q(t,z)+\frac{\partial}{\partial z}Q(t,z)=0, \tag{8}$$

$$Q(t,z)=\Theta(t,z)-\frac{\partial}{\partial z}\Theta(t,z),$$

$$\Theta(0,z)=0,\qquad Q(t,0)=\delta(t)$$

with a point source at $z=0$ as the boundary condition for the function $Q(t,z)$. This model problem is solved explicitly using the Laplace transform (Abramovitz, 1964). In particular,

$$Q(t,z)=\frac{z}{2\sqrt{\pi t^3}}\exp\left\{-\frac{(z-t)^2}{4t}\right\}. \tag{9}$$

The expression for the function $\Theta(t,z)$ is also found explicitly, but contains, in addition to elementary functions, the integral of probabilities. The qualitative behaviour of the quantities $Q(t,z)$ and $\Theta(t,z)$ as functions of t at fixed z is shown in Fig. 3.

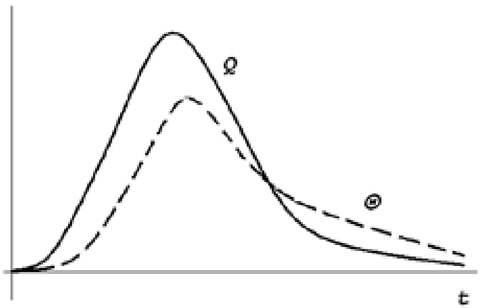

Fig. 3. *Qualitative form of dependence of functions Q (t,z) and Θ (t,z) on the dimensionless time t.*

As the dimensionless depth z increases, the maxima along with the t functions $Q(t,z)$ and $\Theta(t,z)$ decreases and shifts to the right along the axis t. At large z, the position of this maximum of the function $Q(t,z)$ and its value are calculated by the approximate formulas

$$t_{\max}=z(1-3/z+...), \tag{10}$$

$$Q_{\max}=Q(t_{\max},z)=\frac{1}{2\sqrt{\pi z}}\left(1+\frac{9}{4z}+...\right).$$

For what follows, the following integral relations will be needed, which can be obtained both using formula (9) and directly from equations (8):

$$\int_0^\infty Q(t,z)dt=\int_0^\infty\Theta(t,z)dt=1, \tag{11}$$

$$\int_0^\infty tQ(t,z)dt=z,\qquad \int_0^\infty t^2Q(t,z)dt=z^2+2z.$$

An integral representation of the general solution of the problem in the form (7) allows one to study the asymptotics of solutions in various parameters. For example, for large values of time t, the functions Q and Θ on the interval of integration $0<\tau<T$ with respect to τ are almost constant, therefore, for $t\to\infty$

$$q'(t,z) \approx \frac{MV_0}{\sigma} Q\left(\frac{V_0 t}{\sigma}, \frac{z}{\sigma}\right), \qquad \theta'(t,z) \approx \frac{M}{m\sigma} \Theta\left(\frac{V_0 t}{\sigma}, \frac{z}{\sigma}\right),$$

where $M = \int_0^T r(t)dt.$

From this, in particular, it follows that at great depths (more precisely, at $z >> V_0^2 T^2 / \sigma$)

$$\max_t q'(t,z) \approx \frac{MV_0}{2\sqrt{\pi\sigma z}}, \qquad (12)$$

$$\max_t \theta'(t,z) \approx \frac{M}{2\sqrt{\pi\sigma z}}.$$

Thus, at significant times and depths, all details of the behaviour of the initial feed pulse *r(t)* (except for the total volume of additional moisture supplied with the pulse *M)* cease to be significant, and the solution profiles d are determined by the standard functions Q and Θ.

Relations (12) make it possible to express the value of the depth, starting from which the amplitudes of the solutions of the linearised problem become sufficiently small, i.e. do not exceed a particular predetermined threshold value. These asymptotics, however, are not uniform in the remaining parameters of the problem. To get around this problem, it is convenient to monitor not the magnitude and position of the maxima of the solutions, but the behaviour of some integral characteristics of the moisture flow. For this purpose, we define the following values:

$$T_q(z) = \frac{1}{M}\int_0^\infty t q'(t,z)dt, \quad D_q^{cap}(z) = \frac{2\pi}{M}\int_0^\infty (t - T_q(z))^2 dt,$$

$$A_{eff}^{cap}(z) = \frac{M}{\sqrt{D_q^{cap}(z) + Const}}$$

Functions $T_q(z)$ and $D_q^{cap}(z)$ have the meaning of the characteristic time lag pulse infiltration to the depth and the square of its characteristic length at this depth, and $A_{eff}^{cap}(z)$ - the effective flow characteristic amplitude. From the integral representation of Duhamel (7) and relations (11), explicit formulas can be easily derived

$$T_q(z) = T_q(0) + z / V_0, \quad D_q^{cap}(z) = D_q^{cap}(0) + 4\pi\sigma z / V_0^2, \qquad (13)$$

where $T_q(0)$ and $D_q^{cap}(0)$ are the corresponding moments of the feed given on the surface $r(t)$. In the case of a power supply mode specified in the form of a rectangular step (5), $T_q(0) = T/2$ and $D_q^{cap}(0) = \pi T^2 / 6$.

The coefficient 2π in the definition of the integral quantity $D_q^{cap}(z)$ is chosen so that at $z \to \infty$ the effective amplitude of the flow $A_{eff}^{cap}(z)$ asymptotically coincides with its present amplitude by formula (12). In this case, the constant $Const$ can be chosen so that the value $A_{eff}^{cap}(0)$ coincides with the given pulse amplitude on the surface, i.e. with maximum function *r(t)*. For power *R(t)* specified in the form of a step (5) $Const = (1 - \pi/6)T^2$. Since the effective amplitude, in contrast to the present, is determined by the data of the linearised problem of a simple explicit formula , it is helpful for estimating the depth of the effect of unsteadiness surface recharge.

Explicit solution of the infiltration problem in the capillary-free approximation

The model, which is obtained from the formal limit of the Richards equations (2 - 3) at $h \to 0$, i.e. their capillary-free version was proposed in (J.R. Philip, 1954). This model turned out to be useful for substantiating the method for processing experimental data on infiltration, which appeared in earlier work (Green, 1911), named after its authors is called the Green and Ampt model (see Philip, 1969, Egorov, 2003, Chen, 2015). The rigorously mathematically correctness of the passage to the limit for $h \to 0$ in the Richards equations was proved for stationary problems in (Belyaev, 2015, Alt, 1979, Beliaev, 2015, and for non-stationary problems, in Alt, 1984).

In the Green and Ampt model, the filtration equations (2) and the phenomenological formula for the relative permeability $k(\theta)$ are retained in their previous form, and instead of capillary equality (3), the relationship between pressure and water saturation uses the relations

$$0 \le \theta \le 1, \qquad p \ge 0, \qquad (1-\theta)p = 0. \qquad (14)$$

For solutions of equations (2), (15), the plane of independent variables *(t,z)* is divided into previously unknown regions, which correspond to saturated flow zones ($\theta=1$, $p\geq 0$), partially saturated ($0<\theta<1$, $p=0$) and dry ($\theta=0$, $p=0$). At the boundaries of the zones, the pressure continuity condition and the mass conservation law must be satisfied. From equations (2) it follows that in saturated zones the flow q depends only on time, and the pressure is a linear function z. In a dry zone, all the required functions θ, q, p and p are known and are equal to zero. In a partially saturated soil zone p=0, and equations (2) are reduced to one first-order differential equation of the form

$$m\frac{\partial\theta}{\partial t}+K\frac{\partial k(\theta)}{\partial z}=0. \tag{15}$$

Methods for constructing exact solutions of hyperbolic equations have been studied in detail in gas dynamics (Loitsiansky L.G., 1987, Ch. 6, Gurbatov S.N., 2011), as well as in connection with problems of nonlinear sorption (Venitsianov E.V., 1983) and many others. The characteristics $z=z(t)$ of equation (15) are solutions of the ordinary differential equation

$$m\frac{dz}{dt}=Kk'(\theta(t,x)). \tag{16}$$

From equation (15) it follows that along with the characteristics, $d\theta(t,x(t))/dt=0$, therefore, on the characteristics, the water saturation is constant, and the characteristics themselves are straightforward.

Equations (2), (15) can have discontinuous solutions and must be supplemented with conditions wherever there are jumps in water saturation. If the law of motion of a jump on a plane *(t,z)* has the form $z=z(t)$ and the values of water saturation and flow before and after the jump at the moment t are equal θ_0, q_0 and, θ, q respectively, then the law of conservation of mass implies the equality

$$m\frac{dz}{dt}=\frac{q-q_0}{\theta-\theta_0}. \tag{17}$$

For functions $k=k(\theta)$ with a downward convex graph, those jumps for which $\theta<\theta_0$ are not evolutionary, i.e. cannot form due to the evolution of a continuous water saturation profile. To ensure the uniqueness of the solution to the problem, such jumps should be excluded from consideration. Thus, the condition on the jump (17) must be supplemented by an inequality $\theta>\theta_0$. The convexity of the function $k(\theta)$implies that for allowed jumps, i.e. at $\theta>\theta_0$, the rupture velocity is less than the characteristic velocity $Kk'(\theta)/m$ at the points adjacent to the rupture behind but greater than before the jump.

Using the characteristic equation (16) and the condition on discontinuities (17), it is possible to construct various exact solutions of the equations in a partially saturated medium, gluing them with solutions in the saturated and dry zones, if any, and along this path to obtain explicit expressions for the solutions in the entire area.

In contrast to the linearised model for a power supply pulse *R(t)* of an arbitrary shape, it is not possible to derive a unified formula for the general solution of a nonlinear capillary-free problem. It is relatively simple to construct a solution to this problem in the class of functions monotonically decreasing in time *R(t)*. We will construct an exact solution to this problem in an explicit form. The main qualitative conclusion that will be substantiated using this example is as follows. We will show that, despite the absence of capillary dissipation, the water saturation profile in the process of moving downward spreads and approaches the undisturbed state θ_0. Estimates will also be given for the characteristic depth of the influence of the non-stationarity of the external inflow. In this case, stabilisation occurs in a power-law manner in depth and time.

Let the function *R(t)* be continuous and strictly decreasing from the value R_1 at t=0 to the value $R_0<R_1$ at $t=T$. Note that the rectangular pulse (5) is neither continuous nor strictly decreasing, however, its solution can be obtained by limiting the transition of the solutions for the class. By T_1=0 we denote a point in time for which $R(T_1)=K$. If $R_1<K$, then we will assume that T_1=0. Let us also introduce the notation $\theta_1=\left(R(T_1)/K\right)^{1/\beta}$. We define a continuous monotonically decreasing function $\tau=\tau(\theta)$ on an interval $\theta_0\leq\theta\leq\theta_1$ by equality $R(\tau(\theta))=K\theta^{\beta}$. Then $\tau(\theta_0)=T$ and $\tau(\theta_1)=T_1$.

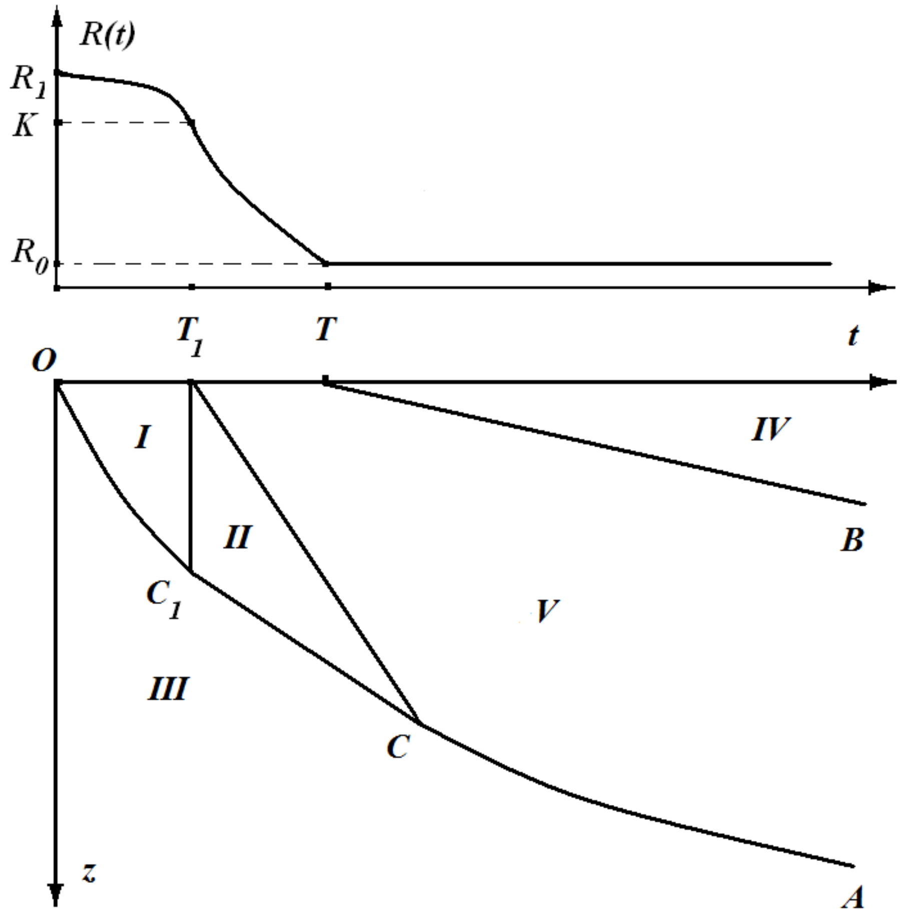

Fig. 4. *Schematic description of the structure of the solution to the problem of capillary infiltration in a pulsed power supply mode.*

The structure of the solution to the problem, constructed by the method of characteristics, is schematically shown in Fig. 4. On this diagram, zones I, II, III, and IV, V are shown, in which the solution is given by expressions of different types. The diagram shows the case of pressure filtration $R_1 > K$. When $R_1 \le K$ zones I and II are absent on the diagram, and points T_1, C_1 and O coincide with the origin . The procedure for constructing these zones and explicit formulas for the sought functions in them is described below.

At the initial moment, an instantaneous change in the inflow from value R_0 to value $R_1 > R_0$ will lead to the appearance of a jump in water saturation from θ_0 to value $\theta_1 > \theta_0$. If at the same time $R_1 \ge K$, then $\theta_1 = 1$, and the soil behind the jump will be water-saturated. The further movement of the jump is shown in the diagram by the curve OC_1CA. Below this curve, i.e. in zone III , the flow remains unperturbed and the solution is given by formulas (4). After the jump, the flow is continuous, and along the ray TB continuously adheres to the undisturbed flow of zone IV. Thus, at each depth, the duration of the infiltration pulse is finite.

Zone I corresponds to pressure infiltration. In it $\theta(t,z)=1$, $q(t,z)=R(t)>K$ and the pressure is positive. The section OC_1 of the jump trajectory, i.e. the lower boundary of this zone is determined by integrating the relationship at the discontinuities (17), in which, $q=R(t)$ and $\theta=1$ should be put. The boundary T_1C_1 separating zones I and II is the vertical interval on which $t=T_1$. It, as well as throughout the zone II $\theta(t,z)=1$, $q(t,z)=K$, $p(t,z)=0$.

Boundaries T_1C and C_1C zones II are straight line segments. The slope of the segment C_1C is found from the relationship at the discontinuities (17), where $q=K$, $q_0=K\theta_0^\beta$, $\theta=1$ and the slop T_1C is found from the equation of characteristics (16), in which $\theta(t,z)\equiv 1$. As a result of simple calculations for the coordinates of a point C, the following formulas are obtained

$$z_C=\frac{\beta M_1}{m\left[(\beta-1)\theta_1^\beta-\beta\theta_1^{\beta-1}\theta_0+\theta_0^\beta\right]},\quad t_C=$$

$$T_1+\frac{M_1}{K\left[(\beta-1)\theta_1^\beta-\beta\theta_1^{\beta-1}\theta_0+\theta_0^\beta\right]},$$

where $M_1=\int_0^{T_1}(R(t)-R_0)dt$.

In zone V, the characteristics of the solution by equality (16) are rays and are given by the equations

$$mz=\beta K\theta^{\beta-1}(t-\tau(\theta)),\quad \theta_0<\theta<\theta_1. \tag{18}$$

Monotonic decrease of the previously defined function $\tau(\theta)$ ensures that this equation can be uniquely solved relative θ. Thereby obtained function $\theta=\theta(t,z)$ is a solution in regarded zone V. Water saturation $\theta(t,z)$ within zone V increases with depth z and decreases with time t. The characteristics begin at the points of the segment T_1T at $z=0$ and fan out without intersecting with each other. The gas dynamics of a solution with such a structure is called Riemann or unloading waves. Rays T_1C and TB are the border zone V and given by the equations (18), in which the water saturation θ takes maximum and minimum values θ_1 and θ_0, respectively, and the value $\tau(\theta)$ - the value $\tau(\theta_1)=T_1$ and $\tau(\theta_0)=T$.

To complete the construction of the solution, it remains to determine the formula for the curvilinear boundary CA separating zones III and V in scheme 4. At each point of this curve, located at depth z, the water saturation abruptly changes from a value $\theta=\theta_0$ to some variable value $\eta=\eta(z)>\theta_0$. For the equation $z=z(t)$ of this curve, it follows from the relations on the jump (17) that

$$m\frac{dz}{dt}=K\frac{\eta^\beta-\theta_0^\beta}{\eta-\theta_0}. \tag{19}$$

Since the quantity η refers to the solution in the zone of the unloading wave V, one can use the equation of characteristics (18) for it and obtain an additional relation

$$mz=\beta K\eta^{\beta-1}(t-\tau(\eta)), \tag{20}$$

binding values z, t and η on the trajectory of the jump CA. Together with the relation (20) the first order differential equation (19) is solved in quadrature with the initial conditions $z(t_C)=z_C$, $\eta(z_C)=\theta_1$. Omitting the calculations, we give an explicit relation between z, t and η on the trajectory of the jump CA:

$$\frac{mz}{\beta}\frac{(\beta-1)\eta^\beta-\beta\theta_0\eta^{\beta-1}+\theta_0^\beta}{\eta^\beta}=M-f(\eta), \tag{21}$$

where $f(\eta)=\int_{\tau(\eta)}^{T}(R(t)-R_0)dt.$

Equalities (20) and (21) define the trajectory of the jump CA in a parametric form, if the quantity η is considered as a parameter varying within $\theta_0<\eta<\theta_1$. The value of the parameter $\eta=\theta_1$ corresponds to the point C of the trajectory, and at, $\eta\to\theta_0$ point (z,t) of the trajectory goes to infinity. Further when $z\to\infty$ one needs to express the magnitude and duration of the jump asymptotically. The auxiliary function $f(\eta)$ from equality (21) at $\eta\to\theta_0$ tends to zero faster than $\eta-\theta_0$. Using this circumstance, from (21) we obtain

$$\frac{\eta(z)-\theta_0}{\theta_0}=\left(\frac{2M}{(\beta-1)mz\theta_0}\right)^{1/2}$$

$$\left[1+\frac{\beta+1}{6}\left(\frac{2M}{(\beta-1)mz\theta_0}\right)^{1/2}+...\right] \qquad (22)$$

Using this relation and equality (20), it is easy to obtain the asymptotic expression for the pulse duration at depth z

$$t_{TB}(z)-t_{OCA}(z)=\frac{\left(2(\beta-1)Mmz\theta_0\right)^{1/2}}{\beta K\theta_0^{\beta}}-\frac{(2\beta-1)M}{3\beta K\theta_0^{\beta}}+..., \qquad (23)$$

where through $t=t_{TB}(z)$ and $t=t_{OCA}(z)$ marked the ray equation TB and the curved trajectory OC_1CA jump respectively. The expression for the function of $\eta(z)$ and $t_{OCA}(z)$ defined by the equations (20) and (21) which, in the general case can not be solved explicitly, but allow to construct asymptotic expansions of the solutions. From (22) and (23) it follows that the magnitude of a jump $\eta(z)-\theta_0$ decreases with increasing depth in inverse proportion to the square root of z at great depths. At the same time, the duration of the infiltration pulse increases in proportion to the same root. It is noteworthy that not only the main terms of these asymptotics, but also the first corrections to them do not depend on the shape of the infiltration pulse $R(t)$, keeping only the dependence on the total additional volume of moisture supplied with the pulse M.

Explicit estimates of the depth of the inþuence of unsteady surface nutrition

In the considered capillary-free problem at an arbitrary depth, the $z>z_C$ amplitude of flux fluctuations in time is equal to

$$A_q^{nl}(z)=\max_t\left(q(t,z)-R_0\right)=K\left(\eta^{\beta}(z)-\theta^{\beta}\right)$$

According to asymptotic formula (22) at $z\to\infty$

$$A_q^{nl}(z)\approx\beta K\vartheta_0^{\beta-1}\left(\eta(z)-\theta_0\right)\approx\beta K\theta^{\beta}\left(\frac{2M}{(\beta-1)mz\theta_0}\right)^{1/2}$$

By analogy with the linearized problem, we introduce into consideration the quantities

$$D_q^{nl}(z)=\left(t_{TB}(z)-t_{OCA}(z)\right)^2/4\approx\frac{(\beta-1)Mmz\theta_0}{2\left(\beta K\theta_0^{\beta}\right)^2},$$

$$A_{eff}^{nl}(z)=\frac{M}{\sqrt{D_q^{nl}(z)+Const}}. \qquad (24)$$

Here, the factor ¼ in the definition $D_q^{nl}(z)$ is selected so that the effective amplitude of the pulse $A_{eff}^{nl}(z)$ t $z\to\infty$ in the main z term coincides with the true amplitude $A_q^{nl}(z)$. The constant $Const$ in this ratio can be chosen so that at the $z=0$ value of the expression on the right side coincides with the given amplitude of the flow on the surface, i.e. with maximum function $R(t)-R_0$. In the situation under consideration $Const=(M/(R_1-R_0))^2-T^2/4$.

Returning to the original problem, in which both nonlinearity and capillary terms are present, we denote by a $D_q(z)$ function equal in order of magnitude to the square of the average pulse duration at depth z. The specific form of this function will be defined later. We assume the growth rate of this function in depth to be in the following form:

$$\frac{d}{dz}D_q(z)=\frac{d}{dz}D_q^{cap}(z)+\frac{d}{dz}D_q^{nl}(z) \qquad (25)$$

This assumption is that the squared pulse duration increases with depth with a velocity that is the sum of the previously found corresponding velocities in the linearised and capillary-free problems. This equality implies that $D_q(z)=D_q^{cap}(z)+D_q^{nl}(z)+Const$

Since the total volume of moisture supplied with an impulse to the depth z is fixed and equal M, the amplitude of the flow in order of magnitude should be

$$A(z)\sim A_{eff}(z)=\frac{M}{\sqrt{D_q(z)}}=\frac{M}{\sqrt{D_q^{cap}(z)+D_q^{nl}(z)+Const}}$$

(26)

The formulas for $D_q^{cap}(z)$ and $D_q^{nl}(z)$ were obtained earlier, and the constant $Const$ in this relation can be chosen so that for $z=0$, the value of the expression on the right-hand side coincides with the given amplitude of the flow on the surface, i.e. with maximum function $R(t)-R_0$. In particular, for a pulse $R(t)$ in the form of a rectangular step (5)

$Const = (9 - 2\pi)T^2/12$. Hypothesis of adding velocities (25) from the reduced to the assumption of approximate equalities of amplitude fluctuations of flow $A(z)$ and effective amplitude $A_{eff}(z)$ as defined by equation (26).

With increasing depth z, the effective amplitude of flux fluctuations $A_{eff}(z)$ decreases and tends to zero. The hypothesis of the addition of the effects of capillary dissipation and nonlinear scattering (25) makes it possible to determine the depth of the influence of unsteady surface feeding $R(t)$ as such a value z, for which the equality holds true

$$A_{eff}(z_\varepsilon) = \varepsilon R_0 \tag{27}$$

where ε is a given small dimensionless number that determines the threshold value of the amplitude of flux fluctuations, below which the flux can be considered steady. The analytical expression for the effective amplitude $A_{eff}(z)$ in this equality is determined by the chain of formulas (26), (13), (24) and algebraic equations (20) and (21). In the case of general position, equations (20) and (21) cannot be solved explicitly , but can be investigated asymptotically. Since the depth of the non-stationary zone z is large for sufficiently small values ε, in equation (27) one can use an approximate expression for the effective amplitude

$$A_{eff}(z) \approx \frac{\beta M R_0}{\left(4\pi\sigma\theta_0^2 m^2 z + (\beta - 1) m M \theta_0 z/2\right)^{1/2}}$$

from where

$$z_\varepsilon \approx \frac{2}{8\pi\sigma\theta_0^2 m^2 + (\beta - 1) M m \theta_0}\left(\frac{\beta M}{\varepsilon}\right)^2$$

Conclusions

1. Two physical mechanisms lead to the stabilisation of the flow in the case of impulse moisture penetration into the soil: capillary dissipation and nonlinear dispersion of the irregularities of the moisture profile.

2. To analyse purely mathematical methods for solving it is advisable to simplify applying the principle of decomposition, namely, to consider the influence of the capillary dissipation and nonlinearity separately and then summarise the results somehow. Taking into account only capillary effects means linearising the Richards equations, and ignoring capillary forces from the Richards equations results in a nonlinear system called the Green-Ampt model in the literature. Both simplified problem models of the pulse and the periodic arrival of moisture in the soil can be solved by explicit formulas.

3. With a pulsed moisture supply, regardless of the pulse shape, the amplitude of solutions at great depths decreases in inverse proportion to the square root of the depth in both the linearised and capillary-free problems. This enables a single explicit formula for the typical depth at which infiltration unsteady flow becomes substantially.

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